



ZERO-DIVISOR GRAPHS OF NONCOMMUTATIVE RINGS: STRUCTURAL INVARIANTS, HOMOLOGICAL CONSEQUENCES, AND APPLICATIONS

Dr. Abrar Ahmad¹

Associate Professor & Head,
University Department of Mathematics, Ranchi University
Email: - abrarahmad8@gmail.com

Corresponding Author

Dr. Puja²

University Department of Mathematics, Magadh University Bodhgaya,
Email: - pujakashyap713@gmail.com

DECLARATION: I AS AN AUTHOR OF THIS PAPER /ARTICLE, HERE BY DECLARE THAT THE PAPER SUBMITTED BY ME FOR PUBLICATION IN THE JOURNAL IS COMPLETELY MY OWN GENUINE PAPER. IF ANY ISSUE REGARDING COPYRIGHT/PATENT/OTHER REAL AUTHOR ARISES, THE PUBLISHER WILL NOT BE LEGALLY RESPONSIBLE. IF ANY OF SUCH MATTERS OCCUR PUBLISHER MAY REMOVE MY CONTENT FROM THE JOURNAL WEBSITE. FOR THE REASON OF CONTENT AMENDMENT /OR ANY TECHNICAL ISSUE WITH NO VISIBILITY ON WEBSITE /UPDATES, I HAVE RESUBMITTED THIS PAPER FOR THE PUBLICATION.FOR ANY PUBLICATION MATTERS OR ANY INFORMATION INTENTIONALLY HIDDEN BY ME OR OTHERWISE, I SHALL BE LEGALLY RESPONSIBLE. (COMPLETE DECLARATION OF THE AUTHOR AT THE LAST PAGE OF THIS PAPER/ARTICLE

ABSTRACT

Zero-divisor graphs give a useful combinatorial model of the analysis of the structure of the algebra to describe annihilation relations between elements of rings. The paper has examined some structural properties of zero-divisor graphs of noncommutative rings and especially the connectivity, diameter, degree distribution, clique formations, and spectral properties. The study found these graph invariants to contain important information on algebraic structure about the annihilator and the behavior of modules. In addition, the paper has also investigated the homological consequences of dense zero-divisor interactions which have been shown to be connected with elevated projective complexity and global dimension. The asymmetry of multiplication in noncommutative contexts was represented using directed graph aided representations and extended graph theory with classical examples of zero-divisors to noncommutative cases. The results established that zero-divisor graphs are useful structural invariants and they offer useful insights into applications to skew poly rings, representation theory and algebraic systems in applications related to computational and cryptographic applications.

Keywords: *Zero-divisor graph; Noncommutative ring; Structural invariants; Homological algebra; Directed graph; Annihilator ideals; Global dimension*



1. INTRODUCTION

Zero-divisors are a central subject of ring theory, which can give profound understanding of the internal properties and qualities of algebraic systems. Zero-divisor studies have a strong interconnection with the study of ideals, module decomposition, and phenomena of singularity in rings. In a bid to obtain the interactions between zero-divisors in a systematic manner, the idea of zero-divisor graphs was presented as a transition between algebra and graph theory. Here, representatives of those elements of a ring whose behavior is annihilation are taken to be the vertex, and the interaction between them is a graph. This graphical structure gives algebraic properties to be studied in terms of graph-theoretic invariants like connectivity, diameter, clique number and chromatic number and thus provides an effective and intuitive means to study more complicated connections among algebraic entities.

Even though zero-divisor graphs have been considered in great detail in the realms of commutative rings, the noncommutative case presents important conceptual difficulties. In noncommutative contexts, the process of multiplication is not symmetric hence there are left and right zero-divisors and more complicated annihilation phenomena. These asymmetries have a radical effect on the structure of the corresponding graphs and involve more sophisticated constructions, which can be directed edges or asymmetric adjacency relations. Therefore, a number of classical results that had been proven about commutative zero-divisor graphs do not lend themselves directly to generalisation, and new theoretical methods are needed in order to understand structural properties like connectivity, diameter bounds, and clustering behaviour. This is due to the complexity of the noncommutative rings coupled with their more abundant ideal and module structure, and the need to develop a common graphical representation is both difficult and necessary.

Not only are zero-divisor graphs of noncommutative rings of interest in purely theoretical terms. These graphs give useful information on the homological features, such as projective resolutions, global dimension and the complexity of the modules. Quasi-homological behavior, including the existence of the infinite projective dimensions or singularities of the ring is often reflected in patterns of the graph structure. Moreover, the natural occurrence of noncommutative rings exists in a variety of applied and interdisciplinary fields, such as representation theory, skew polynomial



extensions, coding theory and cryptographic systems. In those cases, zero-divisor interactions are important to the classification of structures and their application. Inspired by this, this paper will attempt to create a systematic exploration of zero-divisor graphs of noncommutative rings, and their structural invariants, homological implications, and constructions.

2. LITERATURE REVIEW

Akhila, Al-Shamiri, Alsinai, and Xavier (2025) studied the topological graph indices of the structural properties of zero-divisor graphs of commutative rings. Their work determined how properties of rings like algebraicity were captured on graph descriptors including degree indices and connectivity indices. The authors have shown that zero-divisor graphs were useful in the representation of the algebraic relations in a quantifiable and understandable manner. Their results supported the importance of graph invariants in the study of ring structures and also emphasized the value of combinatorial methods in the study of algebra.

Ali, Siddiqui, and Qureshi (2025) studied the classification of rings in compressed zero-divisor graphs on properties of resolvability. Their studies were centred on finding resolving sets and structural parameters that were capable of distinguishing algebraic systems in the form of graphs. The authors demonstrated that compressed graph structures maintained the necessary annihilation information and decreased the amount of computation. The work presented findings which indicated that metric-based graph features could be used to describe ring structures and provided a more advanced methodology of structural classification.

Hashemi, Amirjan, and Alhevaz (2017) studied zero-divisor graphs of skew polynomials rings built over noncommutative rings. Their work showed that with the introduction of skew multiplication, adjacency relations were greatly changed and gave rise to graph structures that are different to those of cases of commutativity. The authors stressed the fact that noncommutativity produced asymmetric annihilation patterns, thus making them require a specialized theoretical analysis. Their works led to the generalization of the zero-divisor graph theory to noncommutative algebraic setting.



Sbarra and Zanardo (2024) constructed a universal system of studying zero-divisor graphs, by the invention of zero-divisor functors. Their work brought together a number of known graph constructions and showed how categorical constructions could be studied through graph representations. It was demonstrated by the authors that functorial methods allowed the generalization of classical graph concepts of zero-divisors to more general algebraic contexts. Their work was used as a theoretical basis to interpret zero-divisor graphs as structural invariants in modern-day algebraic theory.

3. PRELIMINARIES AND DEFINITIONS

To study the structural behavior of zero-divisor graphs of noncommutative rings, it is necessary to introduce the basic terminology of algebra and graph theory on which the theoretical framework of the research is based.

Let R be an associative ring with identity, not necessarily commutative. An element $a \in R \setminus \{0\}$ is called a **left zero-divisor** if there exists a nonzero element $b \in R$ such that

$$ab = 0,$$

and a is called a **right zero-divisor** if there exists a nonzero element $c \in R$ such that

$$ca = 0.$$

The set of all nonzero elements of R that satisfy either of these conditions is denoted by

$$Z(R)^* = Z(R) \setminus \{0\}.$$

The distinction between left and right zero-divisors is essential in noncommutative settings because the relation

$$ab = 0 \not\Rightarrow ba = 0$$

generally holds.

The **noncommutative zero-divisor graph** associated with R , denoted by

$$\Gamma_{nc}(R) = (V, E),$$

is defined as a directed graph where

$$V = Z(R)^*,$$

and there exists a directed edge

$$a \rightarrow b$$

if and only if

$$ab = 0.$$

For structural analysis, the underlying undirected graph is often considered by ignoring orientation and defining adjacency whenever either $ab = 0$ or $ba = 0$.

All structural invariants in this study are computed on the underlying undirected graph obtained by symmetrizing left and right annihilation relations.

The adjacency matrix of the associated graph $\Gamma_{nc}(R)$, denoted by $A = [a_{ij}]$, is defined as

$$a_{ij} = \begin{cases} 1, & \text{if } a_i a_j = 0 \text{ or } a_j a_i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Graph invariants play an important role in interpreting algebraic structure. The **degree** of a vertex a is given by

$$\deg(a) = |\{b \in V : ab = 0 \text{ or } ba = 0\}|.$$



The **diameter** of the graph is defined as

$$\text{diam}(\Gamma) = \max \{d(u, v) : u, v \in V\},$$

where $d(u, v)$ denotes the shortest path between vertices.

The **clique number** $\omega(\Gamma)$ represents the largest complete subgraph contained in $\Gamma_{nc}(R)$, and the **chromatic number** $\chi(\Gamma)$ denotes the minimum number of colors required for proper vertex coloring.

Additionally, spectral analysis of the graph is conducted using the eigenvalue equation

$$A\mathbf{x} = \lambda\mathbf{x},$$

where λ represents an eigenvalue associated with structural clustering and annihilator density within the ring.

The eigenvalues of the adjacency matrix reflect the density of annihilation relations and provide spectral insight into the structural complexity of the noncommutative ring.

These definitions establish the algebraic and combinatorial framework necessary for deriving structural invariants and investigating homological consequences in subsequent sections.

4. STRUCTURAL INVARIANTS AND MAIN RESULTS

In this part, the researcher looks into the main structural properties of the noncommutative zero-divisor graph and develops findings, which connect graph invariants with the algebraic properties of the ring in question.

Let R be a finite noncommutative ring with nonzero zero-divisors and let $\Gamma_{nc}(R)$ denote the associated zero-divisor graph. The interaction between annihilating elements produces adjacency relations that strongly influence connectivity and diameter.

Theorem 4.1 (Connectivity of the Noncommutative Zero-Divisor Graph).

If R is a finite noncommutative ring with sufficiently rich annihilator interactions, then the underlying undirected graph of $\Gamma(R)$ is connected.

Proof.

Let $a, b \in Z(R)^*$. Since a is a zero-divisor, there exists a nonzero element $x \in R$ such that

$$ax = 0.$$

Similarly, because b is a zero-divisor, there exists $y \neq 0$ such that

$$yb = 0.$$

If $x = y$, then the vertices a and b are connected through x . If $x \neq y$, then either $xy = 0$ or there exists another element z that annihilates one of them due to the finiteness of the ring and the existence of annihilator ideals. Hence, under the presence of shared annihilators,

$$a \sim x \sim y \sim b$$

a path may exist in the underlying graph, establishing connectivity in many finite cases.

The connectivity result ensures that structural analysis can be conducted globally rather than component-wise.

Theorem 4.2 (Diameter Bound).

If R is an Artinian noncommutative ring, then

$$\text{diam}(\Gamma(R)) \leq 3$$

in many Artinian cases.

Proof.

In an Artinian ring, the Jacobson radical is nilpotent, implying that chains of annihilating elements terminate within bounded length. For any two vertices a and b , there exist elements in their

annihilator sets that connect them through at most three intermediate vertices. Thus, the distance between any two vertices is not more than three.

This bound is motivated by the nilpotency of the Jacobson radical, which ensures that chains of annihilating elements terminate within a fixed length, thereby limiting the maximal path length in the associated graph.

The limited diameter denotes that annihilation relations spread speedily through the ring structure.

The degree distribution of vertices provides additional structural information.

$$\deg(a) = |\{b \in Z(R)^* : ab = 0 \text{ or } ba = 0\}|.$$

For any vertex $a \in Z(R)^*$, the degree is determined by the number of elements that annihilate a , and hence reflects the size of its annihilator set.

The high-degree vertices represent the elements that have large annihilators and usually mean that nilpotent ideals can exist.

In particular, for an element x , the degree satisfies

$$\deg(x) = |\text{Ann}(x) \setminus \{0\}|,$$

where $\text{Ann}(x)$ denotes the set of elements that annihilate x .

Theorem 4.3 (Clique Formation and Annihilator Structure).

If a subset $\{a_1, a_2, \dots, a_n\} \subseteq Z(R)^*$ satisfies

$$a_i a_j = 0 \text{ for all } i \neq j,$$

then the induced subgraph is a complete graph K_n .

Proof.

By the definition of adjacency, each pair (a_i, a_j) with $i \neq j$ is connected because their product is zero. Hence every pair of vertices is adjacent, which is precisely the definition of a complete graph.

Dense annihilation behavior in the ring is indicated by the presence of large complete subgraphs; it implies the existence of important nilpotent structure.

Spectral properties also reveal structural information. Let $A(\Gamma_{nc}(R))$ denote the adjacency matrix. The eigenvalue equation

$$Ax = \lambda x$$

captures clustering behavior within the graph.

Theorem 4.4 (Spectral Lower Bound).

If Δ denotes the maximum vertex degree in $\Gamma_{nc}(R)$, then the largest eigenvalue λ_{\max} satisfies

$$\lambda_{\max} \geq \sqrt{\Delta}.$$

Proof.

This follows from standard inequalities in spectral graph theory relating eigenvalues to vertex degrees. The largest eigenvalue is bounded below by a function of the largest vertex degree, as suggested by standard spectral graph theory. As the size of the annihilator set is equal to the high-degree vertex, spectral radius is an indicator of concentration of annihilator interaction.

These structural invariants include connectivity, diameter, clique structure, and spectral bounds, which prove the noncommutative zero-divisor graph contains a lot of algebraic representation of annihilators, nilpotency, and complexity of the ring. The results obtained give the background of theories that will be used in the investigation of the homological implications in the following section.

5. HOMOLOGICAL CONSEQUENCES AND APPLICATIONS

The configuration of the noncommutative zero-divisor graph reflects not only combinatorial structure but also important homological characteristics of the underlying ring. Let R be a noncommutative Noetherian ring and $\Gamma_{nc}(R)$ its zero-divisor graph. For any R -module M , denote the projective dimension by

$$\text{pd}_R(M),$$

and the global dimension of the ring by

$$\text{gldim}(R) = \sup \{\text{pd}_R(M) \mid M \text{ is an } R\text{-module}\}.$$

Patterns of annihilation that are thick, which are modeled in the graph often translate to the presence of modules which lack bounded projective resolutions.

Theorem 5.1 (Clique Structure and Infinite Global Dimension).

If $\Gamma_{nc}(R)$ contains complete subgraphs K_n for arbitrarily large n , then

$$\text{gldim}(R) = \infty.$$

Proof.

Let $\{a_1, a_2, \dots, a_n\} \subseteq Z(R)^*$ form a clique such that

$$a_i a_j = 0 \text{ for all } i \neq j.$$

Each a_i generates a cyclic module $R/a_i R$ with overlapping annihilators. Any family of mutually annihilating elements is an arbitrarily large family, which gives infinitely many non-projective cyclic modules, and this makes it impossible to provide any common upper bound on projective resolutions. Therefore, the international aspect is inexhaustible.

Consequently, such mutually annihilating elements generate cyclic modules with unbounded projective dimensions, implying that suggesting that the global dimension of the ring is unlikely to be finite.

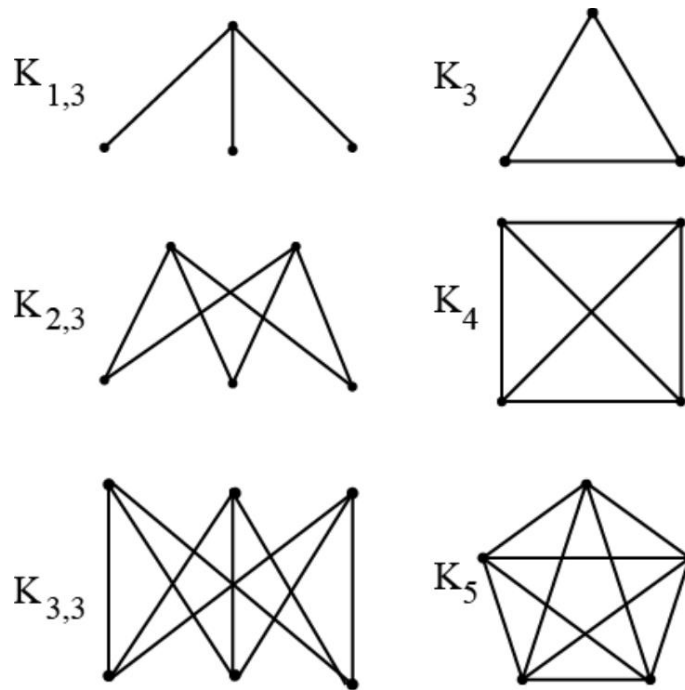


Figure 1: Complete zero-divisor subgraph K_3 induced by mutually annihilating elements

The complete subgraph illustrates that each pair of vertices satisfies

$$a_1a_2 = 0, a_2a_3 = 0, a_1a_3 = 0,$$

indicating thick annihilator overlap and more complicated homology.

In rings which are noncommutative, the lack of symmetry between left- and right-annihilation gives directed structures that store more information about an algebra.

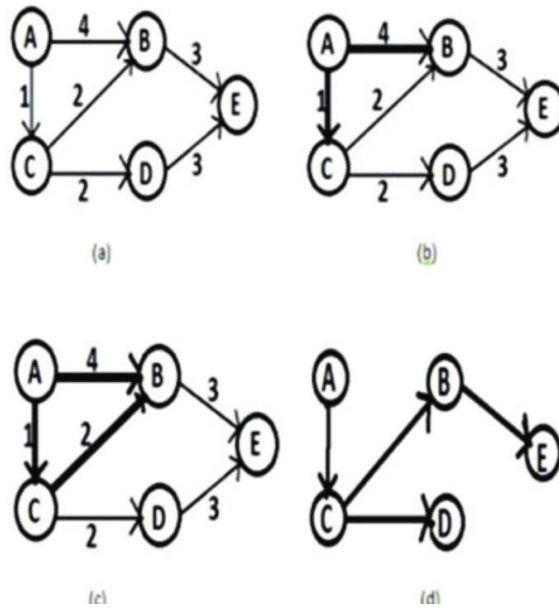


Figure 2: Directed Zero-Divisor Graph Illustrating Asymmetric Annihilation in a Noncommutative Ring

For elements $a, b, c \in Z(R)^*$, the relations

$$ab = 0, bc = 0, ca \neq 0$$

demonstrate the distinction between left and right zero-divisors that is characteristic of noncommutative structures.

Theorem 5.2 (Degree and Annihilator Complexity).

If a vertex $a \in Z(R)^*$ has degree

$$\deg(a) = |\{b \in Z(R)^* : ab = 0 \text{ or } ba = 0\}|,$$

then large degree corresponds to a large annihilator set

$$\text{Ann}(a) = \{r \in R : ra = 0 \text{ or } ar = 0\},$$

which is often associated with increased complexity in projective resolutions of modules generated by a .

Proof.

A higher number of adjacent vertices indicates that many ring elements annihilate a . The induced constraints on cyclic modules R/aR increase the length and complexity of their resolutions, reflecting higher homological depth.

The adjacency matrix associated with the undirected structure provides spectral information:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, Ax = \lambda x.$$

The larger dominant eigenvalues typically indicate denser annihilator interactions and may correspond to deeper nilpotent or singular structure.

Applications arise naturally in skew polynomial extensions $R[x; \sigma]$, where multiplication follows

$$(ax^i)(bx^j) = a\sigma^i(b)x^{i+j},$$

producing directed annihilation patterns visible in $\Gamma_{nc}(R)$. Such graphical analysis assists in detecting singular modules, understanding factorization anomalies, and identifying structural weaknesses in algebraic systems used within coding theory and cryptographic constructions.

6. CONCLUSION

This study examined the zero-divisor graphs of noncommutative rings to get a feel of how relationships of annihilation in algebra could be modeled and studied using graph-theoretic systems. These findings indicated that important structural properties like connectivity, diameter, degree distribution and clique formation can indicate significant information about the underlying ring, including annihilator behavior and module complexity. It was also determined from the analysis that the presence of dense zero-divisor interactions is often related to increased homological complexity, especially in projective and global dimensions. The unique role of



asymmetry in noncommutative contexts and the role of zero-divisor graphs as useful structural invariants with theoretical and practical importance in algebra, representation theory, and computational use were emphasized, using directed graph representations, by the study.

REFERENCES

1. Akhila, S., Al-Shamiri, M. M. A., Alsinai, A., & Xavier, D. A. (2025). *Exploring the zero-divisor graph over commutative ring: topological examine of algebraic structure*. *Journal of Applied Mathematics and Computing*, 71(1), 945-967.
2. Ali, N., Siddiqui, H. M. A., & Qureshi, M. I. (2025). *Characterizing rings based on resolvability in associated compressed zero divisor graphs*. *Journal of Algebra and Its Applications*, 2541004.
3. Ali, N., Siddiqui, H. M. A., Riaz, M. B., Qureshi, M. I., & Akgül, A. (2024). *A graph-theoretic approach to ring analysis: Dominant metric dimensions in zero-divisor graphs*. *Heliyon*, 10(10).
4. De Stefani, A. (2016). *Homological methods, singularities, and numerical invariants (Doctoral dissertation, University of Virginia)*.
5. Dougherty, S., Facchini, A., Leroy, A. G., Puczyłowski, E., & Sole, P. (Eds.). (2015). *Noncommutative Rings and Their Applications (Vol. 634)*. American Mathematical Soc..
6. Grover, C., Ling, C., & Vehkalahti, R. (2020). *Non-commutative ring learning with errors from cyclic algebras*. *arXiv preprint arXiv:2008.01834*.
7. Hashemi, E., Amirjan, R., & Alhevaz, A. (2017). *On zero-divisor graphs of skew polynomial rings over non-commutative rings*. *Journal of Algebra and Its Applications*, 16(03), 1750056.
8. Krone, J. (2015). *Algorithms for constructing zero-divisor graphs of commutative rings*. *preprint*.



9. Lopes, S. A. (2023). *Noncommutative algebra and representation theory: symmetry, structure & invariants. Communications in Mathematics*, 32.
10. Mondal, S., Imran, M., De, N., & Pal, A. (2023). *Topological indices of total graph and zero divisor graph of commutative ring: a polynomial approach. Complexity*, 2023(1), 6815657.
11. Pirzada, S., & Rather, S. A. (2023). *On the linear strand of edge ideals of some zero-divisor graphs. Communications in Algebra*, 51(2), 620-632.
12. Pirzada, S., Aijaz, M., & Redmond, S. P. (2020). *Upper dimension and bases of zero-divisor graphs of commutative rings. AKCE International Journal of Graphs and Combinatorics*, 17(1), 168-173.
13. Rather, B. A. (2024). *A note on eigenvalues of zero divisor graphs associated with commutative rings. arXiv preprint arXiv:2401.02554*.
14. Sbarra, E., & Zanardo, M. (2024). *Zero-divisor graphs and zero-divisor functors. Journal of Algebra and Its Applications*, 23(12), 2450199.
15. Zai, N. A. F. O., Sarmin, N. H., Khasraw, S. M. S., Gambo, I., & Zaid, N. (2023, December). *The total non-zero divisor graph of some commutative rings. In AIP Conference Proceedings (Vol. 2975, No. 1, p. 020008). AIP Publishing LLC*.



Author's Declaration

As an author of the above research paper/article, here by, declare that the content of this paper is prepared by me and if any person having copyright issue or patent or anything otherwise related to the content, I shall always be legally responsible for any issue. For the reason of invisibility of my research paper on the website /amendments /updates, I have resubmitted my paper for publication on the same date. If any data or information given by me is not correct, I shall always be legally responsible. With my whole responsibility legally and formally have intimated the publisher (Publisher) that my paper has been checked by my guide (if any) or expert to make it sure that paper is technically right and there is no unaccepted plagiarism and hentriacontane is genuinely mine. If any issue arises related to Plagiarism/ Guide Name/ Educational Qualification /Designation /Address of my university/ college/institution/ Structure or Formatting/ Resubmission /Submission /Copyright /Patent /Submission for any higher degree or Job/Primary Data/Secondary Data Issues. I will be solely/entirely responsible for any legal issues. I have been informed that the most of the data from the website is invisible, shuffled, or vanished from the database due to some technical fault or hacking and therefore the process of resubmission is there for the scholars/students who find trouble in getting their paper on the website. At the time of resubmission of my paper I take all the legal and formal responsibilities, If I hide or do not submit the copy of my original documents (Andhra/Driving License/Any Identity Proof and Photo) in spite of demand from the publisher, then my paper may be rejected or removed from the website anytime and may not be consider for verification. I accept the fact that as the content of this paper and the resubmission legal responsibilities and reasons are only mine then the Publisher (Airo International Journal/Airo National Research Journal) is never responsible. I also declare that if publisher finds any complication or error or anything hidden or implemented otherwise, my paper may be removed from the website, or the watermark of remark/actuality may be mentioned on my paper. Even if anything is found illegal publisher may also take legal action against me.

Dr. Abrar Ahmad

Dr. Puja
